CONTINUITY

Definition

DEFINITION

Interior point: A function $y = f(x)$ is **continuous at an interior point c** of its domain if

$$\lim_{x \to c} f(x) = f(c).$$

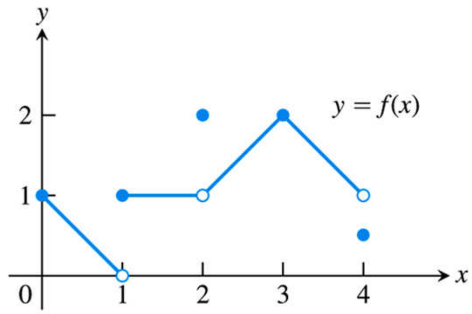
Endpoint: A function $y = f(x)$ is **continuous at a left endpoint a** or is **continuous at a right endpoint b** of its domain if

$$\lim_{x \to a^+} f(x) = f(a) \qquad \text{or} \qquad \lim_{x \to b^-} f(x) = f(b), \quad \text{respectively}.$$

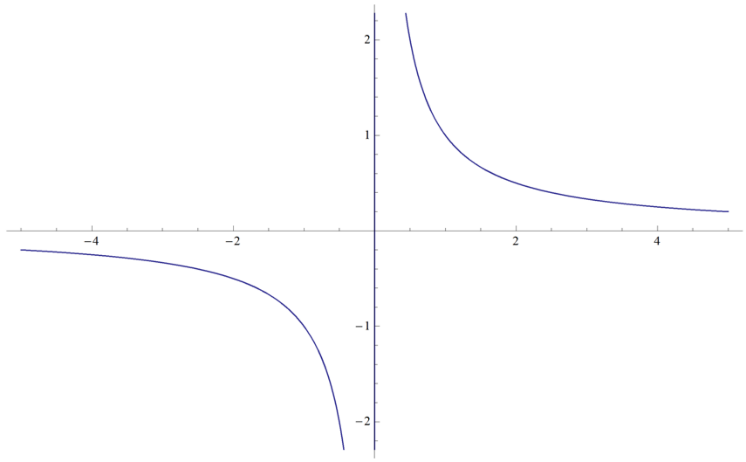
A function is considered to be discontinuous at all points not in its domain

A function that is continuous on its entire domain is called a **continuous function**

Example



Example



Continuity Test

Continuity Test
A function $f(x)$ is continuous at an interior point $x = c$ of its domain if and only if it meets the following three conditions.

1. $f(c)$ exists (c lies in the domain of f).
2. $\lim_{x \to c} f(x)$ exists (f has a limit as $x \to c$).
3. $\lim_{x \to c} f(x) = f(c)$ (the limit equals the function value).

- $f(x) = x$
- $f(x) = |x|$

Proof that Sine is Continuous

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- Sine is defined for all real numbers (1)
- Show that $\lim_{x \to c} \sin(x) = \sin(c)$

  $\lim_{h \to 0} \sin(x+h) = \sin(x)$
- Use the identitiy $\sin(A+B) = \sin A \cos B + \cos A \sin B$

Properties

THEOREM 8—Properties of Continuous Functions If the functions f and g are continuous at $x = c$, then the following combinations are continuous at $x = c$.

1. *Sums:* $f + g$
2. *Differences:* $f - g$
3. *Constant multiples:* $k \cdot f$, for any number k
4. *Products:* $f \cdot g$
5. *Quotients:* f/g, provided $g(c) \neq 0$
6. *Powers:* f^n, n a positive integer
7. *Roots:* $\sqrt[n]{f}$, provided it is defined on an open interval containing c, where n is a positive integer

What does this tell us about polynomials and rational expressions?

Composite Functions

THEOREM 9—Composite of Continuous Functions If f is continuous at c and g is continuous at $f(c)$, then the composite $g \circ f$ is continuous at c.

$$f(x) = \left|x^2 + 2x + 3\right|$$

Limits of Composite Functions

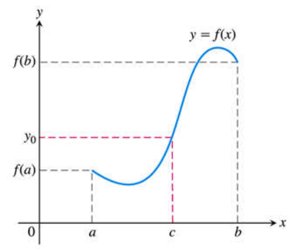
THEOREM 10—Limits of Continuous Functions If g is continuous at the point b and $\lim_{x\to c} f(x) = b$, then

$$\lim_{x\to c} g(f(x)) = g(b) = g(\lim_{x\to c} f(x)).$$

$$\lim_{x\to 3} \ln\left(x^2 - 2x - 2\right)$$

IVT

THEOREM 11—The Intermediate Value Theorem for Continuous Functions If f is a continuous function on a closed interval $[a, b]$, and if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.



Show that $f(x) = x^2 - x - 1$ has a zero in [0,2]