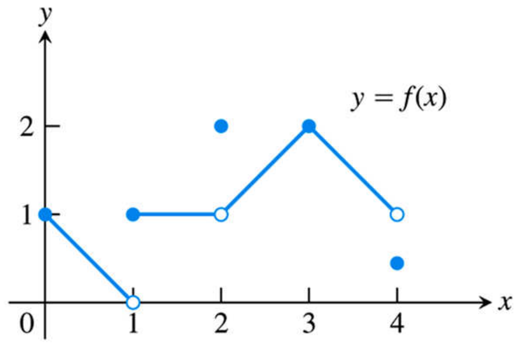
ONE-SIDED LIMITS

Graphically



$a)\lim_{x\to 1^-} f(x)$

$b)\lim_{x\to 1^+} f(x)$

$c)\lim_{x\to 1} f(x)$

$d)\lim_{x\to 2^-} f(x)$

$e)\lim_{x\to 2^+} f(x)$

$f)\lim_{x\to 2} f(x)$

Definition

DEFINITIONS We say that $f(x)$ has **right-hand limit L at x_0**, and write

$$\lim_{x \to x_0^+} f(x) = L \qquad \text{(see Figure 2.28)}$$

if for every number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$x_0 < x < x_0 + \delta \qquad \Rightarrow \qquad |f(x) - L| < \epsilon.$$

We say that f has **left-hand limit L at x_0**, and write

$$\lim_{x \to x_0^-} f(x) = L \qquad \text{(see Figure 2.29)}$$

if for every number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

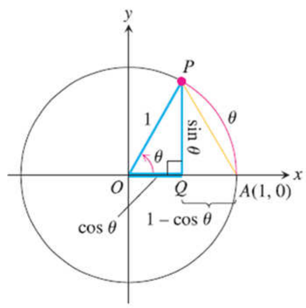
$$x_0 - \delta < x < x_0 \qquad \Rightarrow \qquad |f(x) - L| < \epsilon.$$

Connection to Limit

THEOREM 6 A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x \to c} f(x) = L \quad \Leftrightarrow \quad \lim_{x \to c^-} f(x) = L \quad \text{and} \quad \lim_{x \to c^+} f(x) = L.$$

Useful Limits



$$\sin^2\theta + (1 - \cos\theta)^2 = (AP)^2 \leq \theta^2.$$

Useful Limit

THEOREM 7

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \qquad (\theta \text{ in radians})$$

Proof: Compare areas of the two triangles and the sector

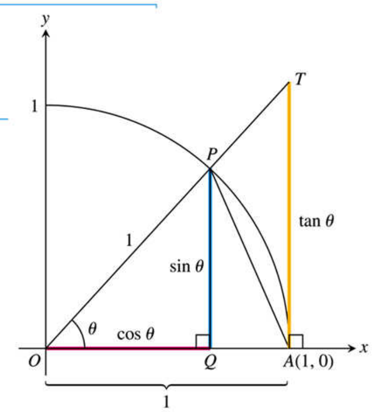


FIGURE 2.33 The figure for the proof of Theorem 7. By definition, $TA/OA = \tan\theta$, but $OA = 1$, so $TA = \tan\theta$.