Math 241
Project 1
Solutions

1) Continuity of Cosine

   a. Prove that $\lim_{\theta \to 0} \cos\theta = 1$, θ in radians

   Since $\cos\theta$ is defined to be the x-coordinate where the terminal side of the angle intersects the unit circle, we know that;

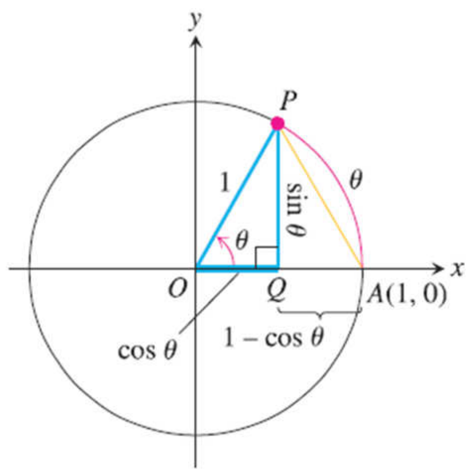
$$-1 \le \cos\theta \le 1.$$

   This means that;

$$0 \le 1 - \cos\theta \le 2.$$

   For θ near zero, by Pythagoras and the fact that the shortest distance between two points is a line segment, we have that;

$$(1-\cos\theta)^2 + \sin^2\theta = AP^2 \le \theta^2 \text{ , } \theta \text{ in radians}$$
$$1 - 2\cos\theta + \cos^2\theta + \sin^2\theta \le \theta^2$$
$$2 - 2\cos\theta \le \theta^2$$
$$1 - \cos\theta \le \frac{\theta^2}{2}$$



   So combining the previous two results, we get that,

$$0 \le 1 - \cos\theta \le \frac{\theta^2}{2}$$

   But we know that;

$$\lim_{\theta \to 0} 0 = 0 \quad \text{and} \quad \lim_{\theta \to 0} \frac{\theta^2}{2} = 0$$

   So, by the Sandwich Theorem, we have that;

$$\lim_{\theta \to 0} (1 - \cos\theta) = 0$$
$$\lim_{\theta \to 0} \cos\theta = 1$$

b. Prove that $\lim\limits_{\theta \to 0} \dfrac{1-\cos\theta}{\theta} = 0$, θ in radians

(Assuming you haven't proved that $\cos\theta$ is continuous.)

$$\lim_{\theta \to 0} \frac{1-\cos\theta}{\theta}\left(\frac{1+\cos\theta}{1+\cos\theta}\right) = \lim_{\theta \to 0} \frac{1-\cos^2\theta}{\theta(1+\cos\theta)} = \lim_{\theta \to 0} \frac{\sin^2\theta}{\theta(1+\cos\theta)}$$

$$= \left(\lim_{\theta \to 0} \frac{\sin\theta}{\theta}\right)\left(\lim_{\theta \to 0} \frac{\sin\theta}{1+\cos\theta}\right) = 0\left(\lim_{\theta \to 0} \frac{\sin\theta}{\theta}\right)$$

$$= 0(1) = 0$$

It remains to be shown that $\lim\limits_{\theta \to 0} \dfrac{\sin\theta}{\theta} = 1$

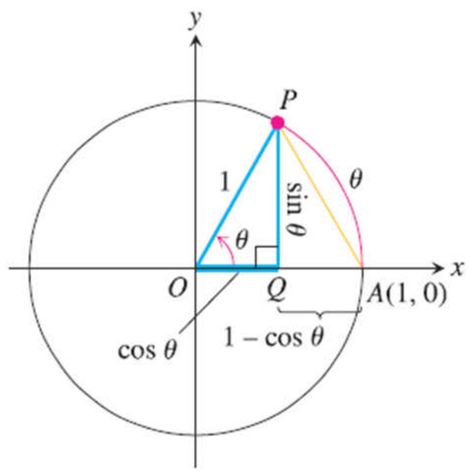
First we will prove that $\lim\limits_{\theta \to 0} \sin\theta = 0$,

From the diagram below and our previous argument;

$$(1-\cos\theta)^2 + \sin^2\theta = AP^2 \le \theta^2 \text{ , } \theta \text{ in radians}$$

$$\sin^2\theta \le \theta^2$$

$$-|\theta| \le \sin\theta \le |\theta|$$



But we know that $\lim\limits_{\theta \to 0} \pm|\theta| = 0$, so by the Sandwich Theorem $\lim\limits_{\theta \to 0} \sin\theta = 0$

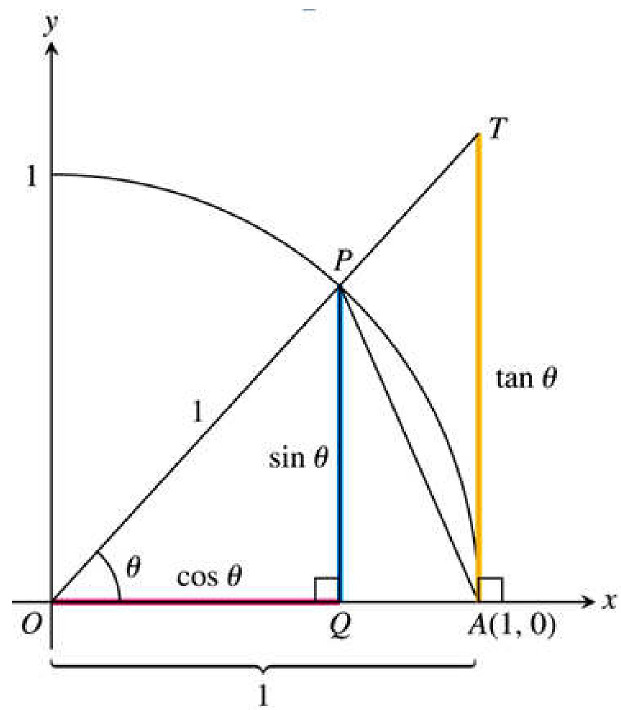
Now we will show that $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

Assuming θ small and $\theta > 0$ .Comparing the areas of the two triangles and the sector in the diagram below, we get the inequality;

$$\frac{1}{2}\sin\theta\cos\theta \le \frac{\theta}{2\pi}\pi(1)^2 \le \frac{1}{2}(1)\tan\theta \text{ , } \theta \text{ in radians}$$

$$\frac{1}{\cos\theta} \le \frac{\theta}{\sin\theta} \le \frac{\cos\theta}{1}$$

$$\frac{1}{\cos\theta} \le \frac{\sin\theta}{\theta} \le 1\cos\theta$$



And we know $\lim_{\theta \to 0} \frac{1}{\cos\theta} = 1 = \lim_{\theta \to 0} \cos\theta$, so by Sandwich Theorem, $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

c. Explain why the unit circle definition of $\cos\theta$ insures that it is defined for all θ

Since we define $\cos\theta$ to be the x-coordinate of the point at which the terminal side of the angle intersects the unit circle, the terminal side cannot pass from inside to outside the circle without intersecting it, so $\cos\theta$ is defined for all θ. Note: This is due to the fact that both $y = \pm\sqrt{1 - x^2}$ are continuous.

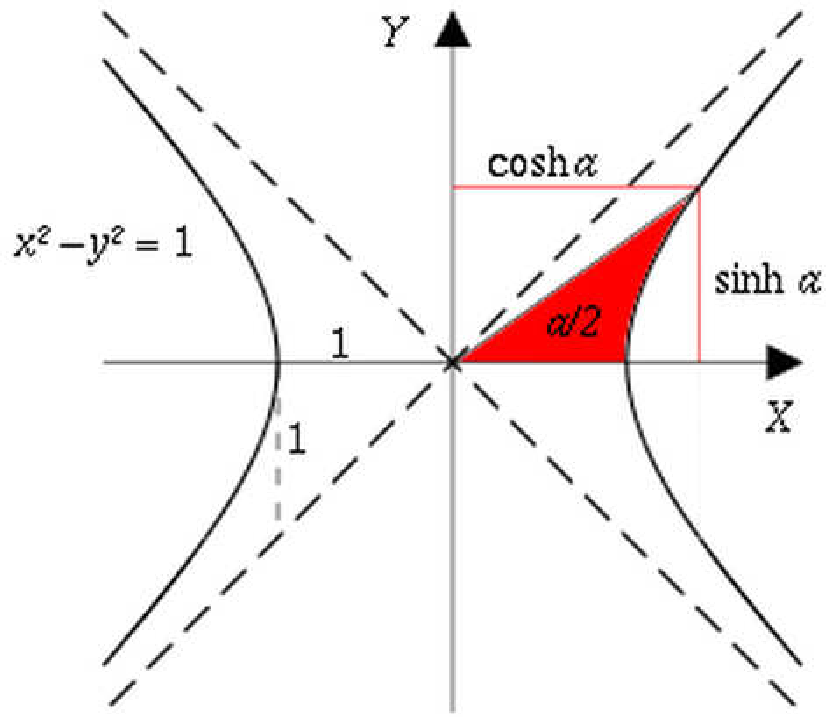
d. Prove that $f(x) = \cos x$ is continuous for all x

$$\lim_{h \to 0} \cos(x+h) = \lim_{h \to 0} \cos(x)\cos(h) - \sin(x)\sin(h)$$
$$= \cos(x)\lim_{h \to 0} \cos(h) - \sin(x)\lim_{h \to 0} \sin(h)$$
$$= \cos(x)(1) - \sin(x)(0)$$
$$= \cos(x)$$

2) Hyperbolic Trigonometric Function

a. Provide a definition of $\sinh(\alpha)$ and $\cosh(\alpha)$ based off of a hyperbola.

We define $\sinh(\alpha)$ and $\cosh(\alpha)$ to be the coordinates of the point on the right arm of the unit hyperbola, $x^2 - y^2 = 1$, that would create and area of $\alpha/2$ as in the diagram below. Note that area below the x-axis is considered to be a negative quantity.



b. Use your previous definition to prove the identity $\cosh^2\alpha - \sinh^2\alpha = 1$.

Since the coordinates on the hyperbola are such that $(x, y) = (\cosh\alpha, \sinh\alpha)$ and the equation of the hyperbola is $x^2 - y^2 = 1$, we get;

$$\cosh^2\alpha - \sinh^2\alpha = x^2 - y^2 = 1$$

c. Provide formulas for $\sinh(\alpha)$ and $\cosh(\alpha)$ based on the natural exponential.

$$\sinh\alpha = \frac{e^{\alpha} - e^{-\alpha}}{2}$$

$$\cosh\alpha = \frac{e^{\alpha} + e^{-\alpha}}{2}$$

d. Use your previous formulas to prove the identity $\cosh^2\alpha - \sinh^2\alpha = 1$.

$$\cosh^2\alpha - \sinh^2\alpha = \left(\frac{e^{\alpha}+e^{-\alpha}}{2}\right)^2 - \left(\frac{e^{\alpha}-e^{-\alpha}}{2}\right)^2 = \frac{e^{2\alpha}+2+e^{-2\alpha}-e^{2\alpha}+2-e^{-2\alpha}}{4} = \frac{4}{4} = 1$$

e. Explain what a catenary curve is.

A catenary curve describes the shape of a rope or chain sagging under the force of gravity alone and supported at its ends. Its equation is based on the hyperbolic cosine.

f. Explain how St. Louis' Gateway Arch relates to this discussion.

The Gateway Arch is an upside down catenary curve. This shape is very stable due to the fact that the internal forces are completely compressive instead of shearing. In other words the downward force of the structure is perpendicular to the ground, so it pushes into the foundation instead of forcing the legs outward.