Instructor: **Grøndahl**

Name: Solutions

MATH 241

Unit 3 Exam

Fall 2017

No Calculators or Notes Allowed

Show all work

Justify All Conclusions

Instructor: **Grøndahl**

1) Determine all the critical points of $y = \ln(x+1) - \tan^{-1} x$

$$\frac{dy}{dx} = \frac{1}{x+1} - \frac{1}{x^2+1}$$

$$= \frac{x^2+1-x-1}{(x+1)(x^2+1)}$$

$$= \frac{x^2-x}{(x+1)(x^2+1)}$$

WHEN IS $\frac{dy}{dx} = 0$?

$$\frac{x(x-1)}{(x+1)(x^2+1)} = 0$$

$$x(x-1) = 0$$

$$x = 0, 1$$

WHEN IS $\frac{dy}{dx}$ UNDEFINED?

$$(x+1)(x^2+1) = 0$$

$$x+1 = 0$$

$$x = -1$$

2) Prove that the function $f(x)=x^4+3x+1$ has exactly one zero on the interval $[-2,-1]$

$f(-2) = 16-6+1 = 11 > 0$

$f(-1) = 1-3+1 = -3 < 0$

SO BY IVT THERE EXISTS $x \in [-2,-1]$

WHERE $f(x)=0$

BUT

$f'(x) = 4x^3+3$

WHICH IS ZERO WHEN

$4x^3+3=0$

$x^3 = -3/4$

$x = \sqrt[3]{3/4}$

WHICH IS NOT IN $[-2,-1]$

SO $f(x)$ HAS ONLY 1 ZERO IN $[-2,-1]$

OR ELSE MVT SAYS THERE

WOULD BE $f'(x)=0$ IN $[-2,-1]$

3) List the open intervals where $f(x)$ is increasing and the open intervals where it is decreasing given that its derivative is the following.

$$f'(x) = (\sin x - 1)(2\cos x + 1)\ ,\ \ 0 \le x \le 2\pi$$

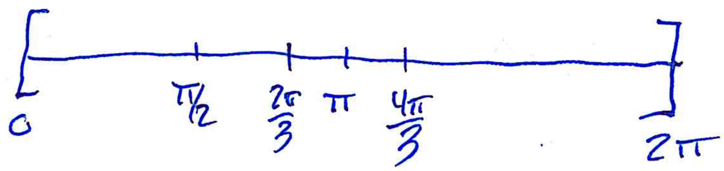
$$(\sin x - 1)(2\cos x + 1) = 0$$

$\sin x - 1 = 0$ $\qquad$ $2\cos x + 1 = 0$

$\sin x = 1$ $\qquad$ $2\cos x = -1$

$x = \frac{\pi}{2}.$ $\qquad$ $\cos x = -1/2$

$\qquad\qquad$ $x = \frac{2\pi}{3}, \frac{4\pi}{3}$



(−) 0 (−) $\quad$ $\sin x - 1$

(+) 0 − 0 (+) $\quad$ $2\cos x + 1$

(−) 0 (−) 0 (+) 0 (−) $\quad$ $(\sin x - 1)(2\cos x + 1)$

Inc: $\left(\frac{2\pi}{3}, \frac{4\pi}{3}\right)$

Dec: $(0, \pi/2)$, $\cancel{(\pi/2}$ $(\pi/2, 2\pi/3)$, $\left(\frac{4\pi}{3}, 2\pi\right)$

4) Given the function $f(x) = x\sqrt{8-x^2}$

a. List all open intervals of increase and decrease

$$f'(x) = \sqrt{8-x^2} + \frac{1}{2}(8-x^2)^{-1/2}(-2x)(x)$$

$$= \sqrt{8-x^2} - \frac{x^2}{\sqrt{8-x^2}}$$

$$= \frac{8-x^2-x^2}{\sqrt{8-x^2}}$$

$$= \frac{8-2x^2}{\sqrt{8-x^2}}$$

$f'(x) = 0$ $f'(x)$ UND

$8-2x^2=0$ $8-x^2=0$

$x = \pm 2$ $x = \pm 2\sqrt{2}$

$-2\sqrt{2}$		-2		0		2		$2\sqrt{2}$	
	$(-)$	0		$(+)$		0	$(-)$		$f'(x)$

Dec: $(-2\sqrt{2}, -2)$
$(2, 2\sqrt{2})$

Inc: $(-2, 2)$

b. Find all local and absolute extrema

Local

Min: $(-2,-4)$, $(2\sqrt{2}, 0)$
Max: $(-2\sqrt{2}, 0)$, $(2, 4)$

Absolute

Min: $(-2,-4)$
Max: $(2,4)$

c. List all open intervals of concave up and concave down

$$f'(x) = \frac{8-2x^2}{\sqrt{8-x^2}}$$

$$f''(x) = \frac{-4x\sqrt{8-x^2} - (8-2x^2)\frac{1}{2}(8-x^2)^{-1/2}(-2x)}{8-x^2}$$

$$= \frac{-4x\sqrt{8-x^2} + \frac{8x-2x^3}{\sqrt{8-x^2}}}{8-x^2}$$

$$= \frac{-32x+4x^2+8x-2x^3}{(8-x^2)^{3/2}} = 0$$

$$-2x^3+4x^2-24x=0$$

$$-2x(x^2-2x+12)=0$$

$$x=0$$

$[-2\sqrt{2}, 2\sqrt{2}]$: $+$ on $(-2\sqrt{2}, 0)$, $-$ on $(0, 2\sqrt{2})$

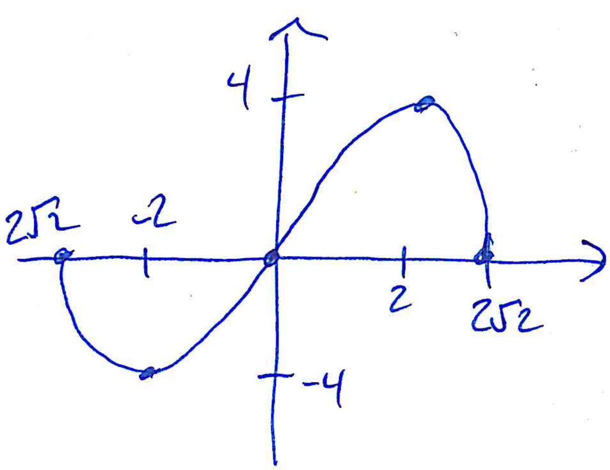
CONCAVE UP: $(-2\sqrt{2}, 0)$

CONCAVE DOWN $(0, 2\sqrt{2})$

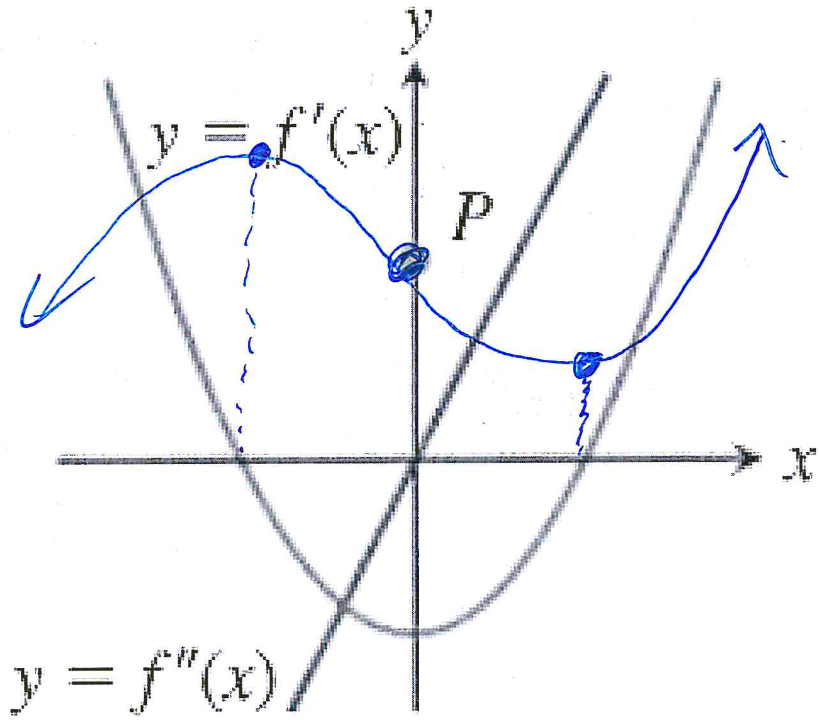
d. Find any inflection points

$(0,0)$

e. Sketch the graph



5) Sketch $y = f(x)$ on the graph below.



6) The intensity of illumination at any point from a light source is proportional to the square of the reciprocal of the distance between the point and the light source. Two lights, one having an intensity eight times the other, are 6m apart. How far from the stronger light is the total illumination least?

$$I(x) = \frac{8k}{x^2} + \frac{k}{(6-x)^2}$$

$$\frac{dI}{dx} = \frac{-16k}{x^3} + \frac{2k}{(6-x)^3} = 0$$

$$\frac{2k}{(6-x)^3} = \frac{16k}{x^3}$$

$$2kx^3 = 16k(6-x)^3$$

$$x^3 = 8(6-x)^3$$

$$x = 2(6-x)$$

$$0 = 12 - 2x - x$$

$$x = 4m$$

7) Use one iteration of Newton's Method with an initial guess of $x = 2$ to approximate a solution to $f(x) = x^2 - 2$

$$x_0 = 2$$

$$f(x_0) = f(2) = 4 - 2 = 2$$

$$f'(x) = 2x$$

$$f'(x_0) = f'(2) = 4$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 2 - \frac{2}{4}$$

$$= 1.5$$

8) Evaluate $\int \left(\frac{1}{x} + \frac{5}{1+x^2} \right) dx$

$$= \ln|x| + 5\tan^{-1}x + C$$