Guide to Series Arguments

When arguing convergence or divergence by the following techniques, be sure to at least include all the bulleted information.

Geometric Series:

$$\sum_{n=4}^{\infty} \frac{2^n}{3^{n+1}}$$

- Convergent geometric series
- $|r| = \left|\frac{2}{3}\right| = \frac{2}{3} < 1$

Telescoping Series:

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

- $S_k = 1 - \frac{1}{k+1}$
- $\lim_{k\to\infty} \left(1 - \frac{1}{k+1}\right) = 1$
- By definition $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$ converges to 1

N-th Term Test:

$$\sum_{n=1}^{\infty} \frac{n}{n+1}$$

- $\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+1/n} = 1 \neq 0$
- $\sum_{n=1}^{\infty} \frac{n}{n+1}$ divergent by nth term test

Integral Test:

$$\sum_{n=1}^{\infty}\frac{1}{n\left(1+\ln^2 n\right)}$$

- $f(x)=\dfrac{1}{x\left(1+\ln^2 x\right)}$ is positive, decreasing, and continuous
- $\displaystyle\int_1^{\infty}\frac{dx}{x\left(1+\ln^2 x\right)}=\lim_{b\to\infty}\int_1^{b}\frac{dx}{x\left(1+\ln^2 x\right)}$ $\quad u=\ln x,\ du=\dfrac{1}{x}dx$

  $\displaystyle=\lim_{b\to\infty}\int_0^{\ln b}\frac{du}{1+u^2}=\lim_{b\to\infty}\left(\tan^{-1}(\ln b)-\tan^{-1}0\right)=\frac{\pi}{2}$
- $\displaystyle\int_1^{\infty}\frac{dx}{x\left(1+\ln^2 x\right)}$ converges, so $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n\left(1+\ln^2 n\right)}$ converges by integral test.

P-Series:

$$\sum_{n=2}^{\infty}\frac{1}{\sqrt{n}}$$

- $\displaystyle\sum_{n=2}^{\infty}\frac{1}{\sqrt{n}}$ divergent p-series $p=\dfrac{1}{2}\le 1$

DCT:

$$\sum_{n=2}^{\infty}\frac{1}{n2^n}$$

- $0\le\dfrac{1}{n2^n}\le\dfrac{1}{2^n}$
- $\displaystyle\sum_{n=2}^{\infty}\frac{1}{2^n}$ is a convergent geometric series $|r|=\left|\dfrac{1}{2}\right|<1$
- So, $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n2^n}$ converges by DCT

LCT:

$$\sum_{n=2}^{\infty}\frac{1}{n^2-1}$$

- $\frac{1}{n^2}$ and $\frac{1}{n^2-1}$ are positive
- $\lim_{n\to\infty}\frac{\frac{1}{n^2-1}}{\frac{1}{n^2}}=\lim_{n\to\infty}\frac{n^2}{n^2-1}=\lim_{n\to\infty}\frac{1}{1-1/n^2}=1>0$
- And $\sum_{n=2}^{\infty}\frac{1}{n^2}$ is a conv. p-series $p=2>1$
- So, $\sum_{n=2}^{\infty}\frac{1}{n^2-1}$ conv. by LCT

Ratio Test:

$$\sum_{n=2}^{\infty}\frac{2^n}{n!}$$

- $\lim_{n\to\infty}\left|\frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}}\right|=\lim_{n\to\infty}\frac{2}{n+1}=0<1$
- So $\sum_{n=2}^{\infty}\frac{2^n}{n!}$ converges by ratio test

Root Test:

$$\sum_{n=1}^{\infty}\left(\frac{\ln n}{n}\right)^n$$

- $\lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{\ln n}{n}\right)^n\right|}=\lim_{n\to\infty}\frac{\ln n}{n}=0<1$

  (Thm. 5)
- $\sum_{n=1}^{\infty}\left(\frac{\ln n}{n}\right)^n$ converges by root test

AST:

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

- $\frac{1}{n^2}>0$
- $\frac{1}{n^2}$ is nonincreasing
- $\lim_{n\to\infty}\frac{1}{n^2}=0$
- So, $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$ converges by AST

ACT:

$$\sum_{n=1}^{\infty}\frac{\cos n}{n^2}$$

- $\sum_{n=1}^{\infty}\left|\frac{\cos n}{n^2}\right|=\sum_{n=1}^{\infty}\frac{|\cos n|}{n^2}$
  - $0\le\frac{|\cos n|}{n^2}\le\frac{1}{n^2}$
  - $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent p-series $p=2>1$
  - So, $\sum_{n=1}^{\infty}\left|\frac{\cos n}{n^2}\right|$ converges by DCT
- So, $\sum_{n=1}^{\infty}\frac{\cos n}{n^2}$ converges by ACT