

We know that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges because it is a p-series with $p = 2 > 1$. In other words, $\sum_{n=1}^{\infty} \frac{1}{n^2} = S$.

What if we know want to know what S actually is? Well we could approximate the value by adding up a bunch of terms and since we p-series are convergent by integral test we know that

$$\int_{N+1}^{\infty} \frac{1}{x^2} dx \leq R_N \leq \int_N^{\infty} \frac{1}{x^2} dx$$

Say we were interested in approximating the sum with an error less than 10^{-8} . That would mean that we would want

$$R_N \leq \int_N^{\infty} \frac{1}{x^2} dx < 10^{-8}$$

So let's solve this inequality for N , so we know how many terms to add up to get the desired error.

$$\int_N^{\infty} \frac{1}{x^2} dx < 10^{-8}$$

$$\lim_{b \rightarrow \infty} \int_N^b \frac{1}{x^2} dx < 10^{-8}$$

$$\lim_{b \rightarrow \infty} \left[\frac{-1}{x} \right]_N^b < 10^{-8}$$

$$\lim_{b \rightarrow \infty} \left[\frac{-1}{b} + \frac{1}{N} \right] < 10^{-8}$$

$$0 + \frac{1}{N} < 10^{-8}$$

$$10^8 < N$$

So we would need to add up more than 100,000,000 terms to guarantee our estimate is not off by more than 0.00000001

Here is what Desmos gives for the approximation



The image shows a Desmos calculator interface. On the left, the expression $\sum_{n=1}^{100000000} \frac{1}{n^2}$ is entered. On the right, the result is displayed as 1.64493405783. There is a small 'x' icon in the top right corner of the calculator area.

This particular sum has been famously found to sum to $\frac{\pi^2}{6}$. You may want to check to see if our approximation stays within our error bound.