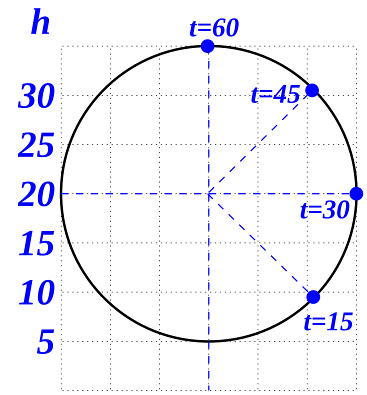
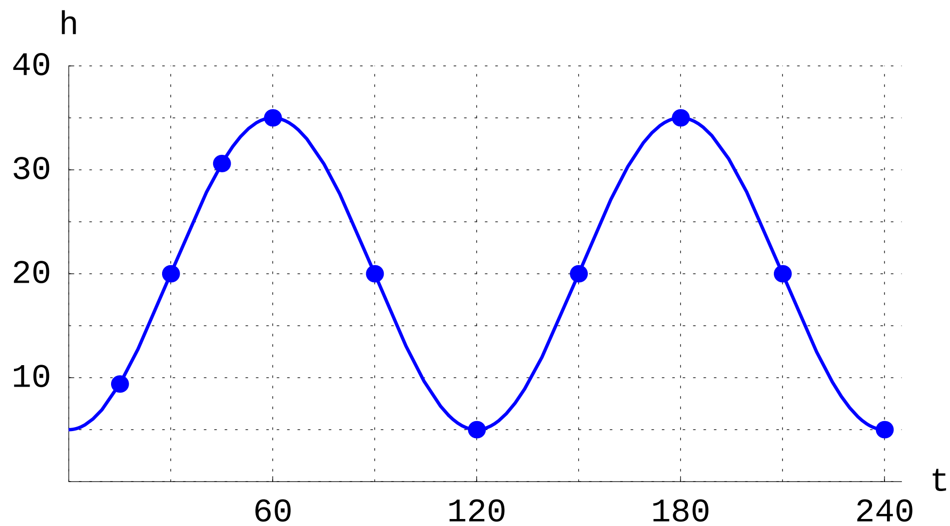
Section 7.1 – Introduction to Periodic Functions

Preliminary Example. The Brown County Ferris Wheel has diameter 30 meters and completes one full revolution every two minutes. When you are at the lowest point on the wheel, you are still 5 meters above the ground. Assuming you board the ride at $t = 0$ seconds, sketch a graph of your height, $h = f(t)$, as a function of time.





Definition. A function f is called *periodic* if its output values repeat at regular intervals. Graphically, this means that if the graph of f is shifted horizontally by p units, the new graph is identical to the original. Given a periodic function f:

1. The *period* is the horizontal distance that it takes for the graph to complete one full cycle. That is, if p is the period, then $f(t + p) = f(t)$.
2. The *midline* is the horizontal line midway between the function's maximum and minimum output values.
3. The *amplitude* is the vertical distance between the function's maximum value and the midline.

Example 1. What are the amplitude, midline and period of the function $h = f(t)$ from the preliminary example?

Period = 120 seconds

Midline: $h = 20$ meters

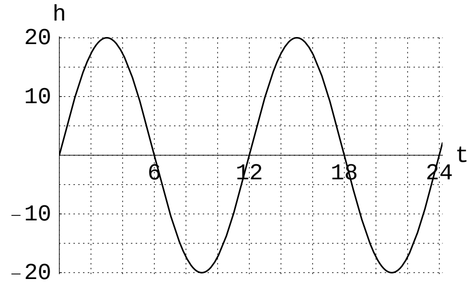
Amplitude = 15 meters

Examples and Exercises

1. The function given below models the height, h, in feet, of the tide above (or below) mean sea level t hours after midnight.

   (a) Is the tide rising or falling at 1:00 a.m.?

   The tide is rising because the graph is increasing at $t = 1$.

   (b) When does low tide occur?

   Low tide occurs at 9 a.m. and again at 9 p.m. because the graph achieves its minimum output value at $t = 9$ and $t = 21$.

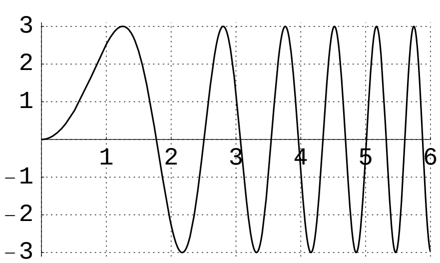
   (c) What is the amplitude of the function? Give a practical interpretation of your answer.

   The amplitude of the function is 20 feet. In this problem, the amplitude represents the maximum amount that the tide deviates from mean sea level.

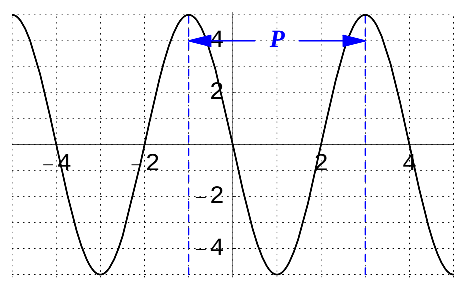
   (d) What is the midline of the function? Give a practical interpretation of your answer.

   The midline of the function is the line $h = 0$. This represents the height of the water when the tide is at mean sea level.

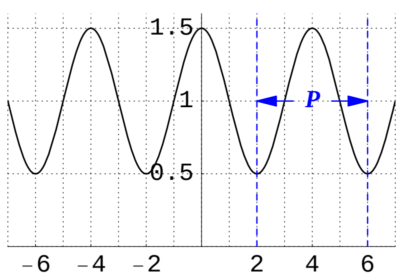
2. Which of the following functions are periodic? For those that are, what is the period?



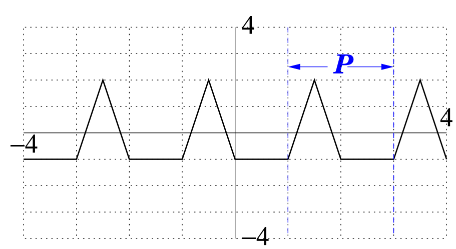
not periodic



periodic, period = 4



periodic, period = 4

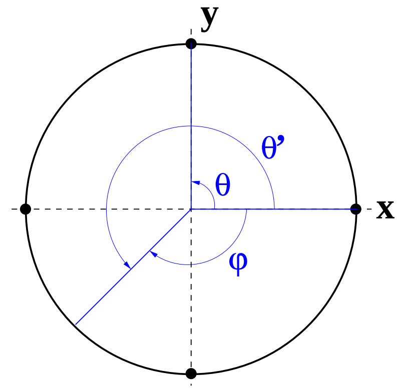


periodic, period = 2

Section 7.2 – The Sine and Cosine Functions

Angle Measurement in Circles

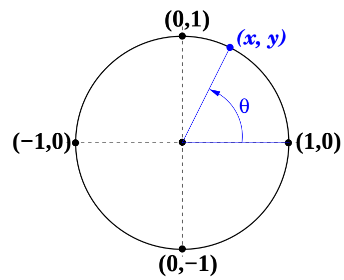
- Angles start from the positive x-axis.
- Counterclockwise defined to be positive.



$$\begin{aligned}\theta &= 90^\circ \\ \theta' &= 180^\circ + 45^\circ = 225^\circ \\ \phi &= -135^\circ\end{aligned}$$

Definition. The *unit circle* is the term used to describe a circle that has its center at the origin and has radius equal to 1. The *cosine* and *sine* functions are then defined as described below.

$$\begin{aligned}\cos\theta &= x \\ \sin\theta &= y\end{aligned}$$

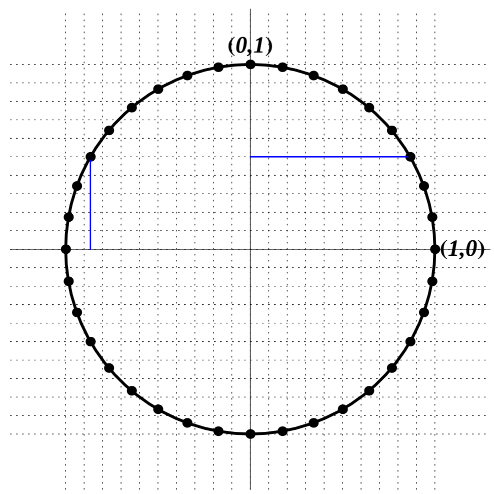


Example 1. On the unit circle to the right, the angles 10°, 20°, 30°, etc., are indicated by black dots on the circle. Use this diagram to estimate each of the following:

(a) $\cos(30^\circ) =$ 0.87

(b) $\sin(150^\circ) =$ 0.5

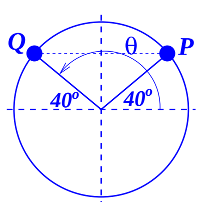
(c) $\cos(270^\circ) =$ 0

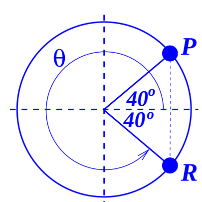


Example 2.

(a) Find an angle θ between 0° and 360° that has the same sine as 40°.

(b) Find an angle θ between 0° and 360° that has the same cosine as 40°.





(a) P and Q have the same y-coordinate, so $\sin 40^\circ = \sin\theta$. Therefore, our answer is

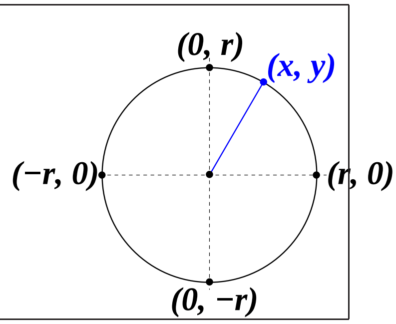
$$\theta = 180^\circ - 40^\circ = 140^\circ.$$

(b) P and R have the same x-coordinate, so $\cos 40^\circ = \cos\theta$. Therefore, our answer is

$$\theta = 360^\circ - 40^\circ = 320^\circ.$$

Theorem. Consider a circle of radius r centered at the origin. Then the x and y coordinates of a point on this circle are given by the following formulas:

$$x = r\cos\theta$$
$$y = r\sin\theta$$



Examples and Exercises

1. Use the unit circle to the right to estimate each of the following quantities to the nearest 0.05 of a unit.

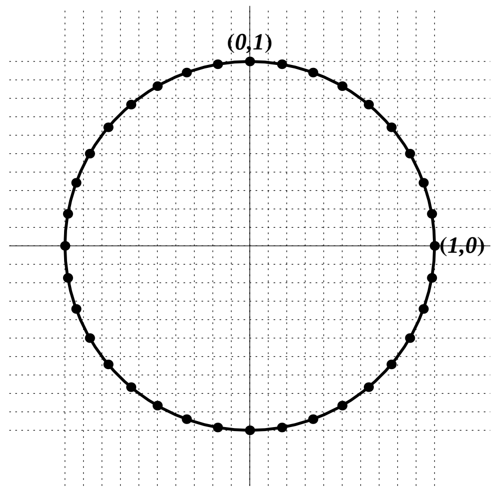
   (a) $\sin(90°) =$ 1 (b) $\cos(90°) =$ 0

   (c) $\sin(180°) =$ 0 (d) $\cos(180°) =$ −1

   (e) $\cos(45°) =$ 0.7 (f) $\sin(-90°) =$ −1

   (g) $\cos(70°) =$ 0.35 (h) $\sin(190°) =$ −0.2

   (i) $\sin(110°) =$ 0.9 (j) $\cos(110°) =$ −0.35



2. For each of the following, fill in the blank with an angle between 0° and 360°, different from the first one, that makes the statement true.

   (a) $\sin(20°) = \sin($160°$)$ (b) $\sin(70°) = \sin($110°$)$ (c) $\sin(225°) = \sin($315°$)$

   (d) $\cos(20°) = \cos($340°$)$ (e) $\cos(70°) = \cos($290°$)$ (f) $\cos(225°) = \cos($135°$)$

3. Given to the right is a unit circle. Fill in the blanks with the correct answer in terms of a or b.

   (a) $\sin(\theta + 360°) =$ b

   (b) $\sin(\theta + 180°) =$ $-b$

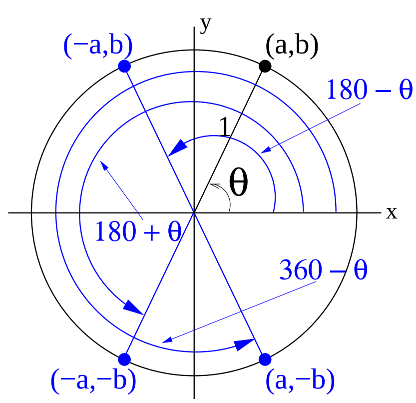
   (c) $\cos(180° - \theta) =$ $-a$

   (d) $\sin(180° - \theta) =$ b

   (e) $\cos(360° - \theta) =$ a

   (f) $\sin(360° - \theta) =$ $-b$

   (g) $\sin(90° - \theta) =$ a



4. Use your calculator to find the coordinates of the point P at the given angle on a circle of radius 4 centered at the origin.

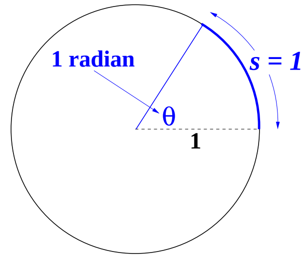
(a) 70° (b) 255°

(a) We have $x = r\cos\theta = 4\cos 70^\circ \approx 1.368$ and $y = r\sin\theta = 4\sin 70^\circ \approx 3.759$. Therefore, our final answer is $(1.368, 3.759)$.

(b) We have $x = r\cos\theta = 4\cos 255^\circ \approx -1.035$ and $y = r\sin\theta = 4\sin 255^\circ \approx -3.864$. Therefore, our final answer is $(-1.035, -3.864)$.

Section 7.3 – Radians and Arc Length

Definition. An angle of 1 radian is defined to be the angle, in the counterclockwise direction, at the center of a unit circle which spans an arc of length 1.



Conversion Factors:

Degrees $\times \frac{\pi}{180}$ $\longrightarrow$ Radians

Radians $\times \frac{180}{\pi}$ $\longrightarrow$ Degrees

Note: Since the length around the entire circle is $2\pi \cdot 1 = 2\pi$, we see that

$$360^\circ = 2\pi \text{ radians.}$$

Example 2. Convert each of the following angles from radians to degrees or from degrees to radians. An angle measure is assumed to be in radians if the degree symbol is not indicated after it.

(a) 30° (b) $\frac{3\pi}{2}$ (c) 1.4

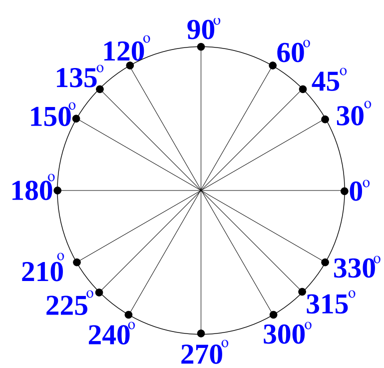
(a) $30^\circ \times \frac{\pi}{180^\circ} = \frac{30\pi}{180} = \frac{\pi}{6}$.

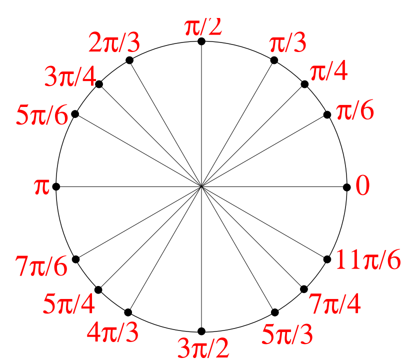
(b) $\frac{3\pi}{2} \times \frac{180^\circ}{\pi} = 3 \cdot 90^\circ = 270^\circ$.

(c) $1.4 \times \frac{180^\circ}{\pi} = \left(\frac{252}{\pi}\right)^\circ \approx 80.2^\circ$.

Example 3. On the unit circle to the right, label the indicated "common" angles with their degree and radian measures.

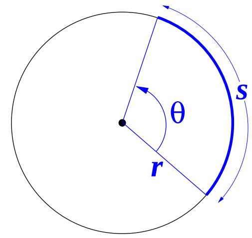
Note: To the right, blue numbers indicate degree measures and red numbers indicate radian measures.





Theorem. The arc length, s, spanned in a circle of radius r by an angle of θ radians, $0 \leq \theta \leq 2\pi$, is given by

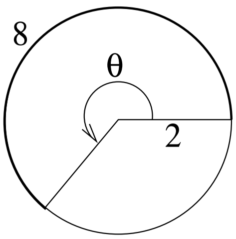
$$s = r\theta$$

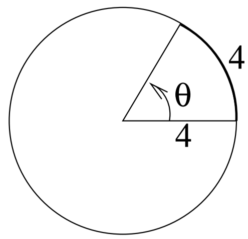


Note: θ must be in radians in order for this formula to work!

Examples and Exercises

1. In the pictures below, you are given the radius of a circle and the length of a circular arc cut off by an angle θ. Find the degree and radian measure of θ.

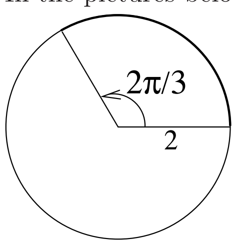


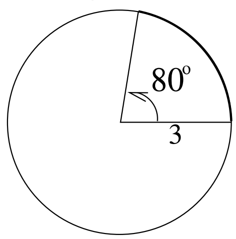


Left Circle: We are given that $s = 8$ and $r = 2$. Therefore, $s = r\theta \implies 8 = 2\theta$, which means that $\theta = 8/2 = 4$ radians, or about $229.2°$.

Right Circle: We are given that $s = 4$ and $r = 4$. Therefore, $s = r\theta \implies 4 = 4\theta$, which means that $\theta = 4/4 = 1$ radian, or about $57.3°$.

2. In the pictures below, find the length of the arc cut off by each angle.





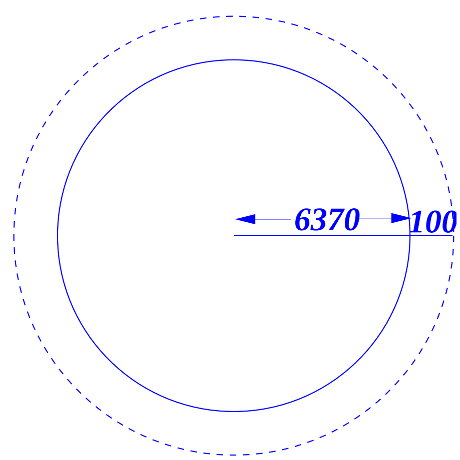
Left Circle: We are given that $r = 2$ and $\theta = 2\pi/3$. Therefore, $s = r\theta = 2 \cdot (2\pi/3)$, which means that $s = 4\pi/3$ units.

Right Circle: We are given that $r = 3$ and $\theta = 80°$. Converting to radians, we have $\theta = 80 \cdot (\pi/180) = 4\pi/9$ radians. Therefore, $s = r\theta = 3 \cdot (4\pi/9)$, so $s = 4\pi/3$ units.

3. A satellite orbiting the earth in a circular path stays at a constant altitude of 100 kilometers throughout its orbit. Given that the radius of the earth is 6370 kilometers, find the distance that the satellite travels in completing 70% of one complete orbit.

First, note that, in the diagram to the right, the inner solid circle denotes the surface of the earth, while the outer dotted circle denotes the path of the satellite in its circular orbit. The radius of the orbital path is $r = 6370 + 100 = 6470$ kilometers. Also, since there are 2π total radians in the central angle of an entire circle, the length of the entire orbital path of the satellite is

$$s = r\theta = 6470 \cdot 2\pi = 12940\pi \text{ kilometers.}$$



Therefore, the distance traveled by the satellite in completing 70% of its orbit is

$$0.7(12940\pi) \approx 28457 \text{ kilometers,}$$

which is our final answer.

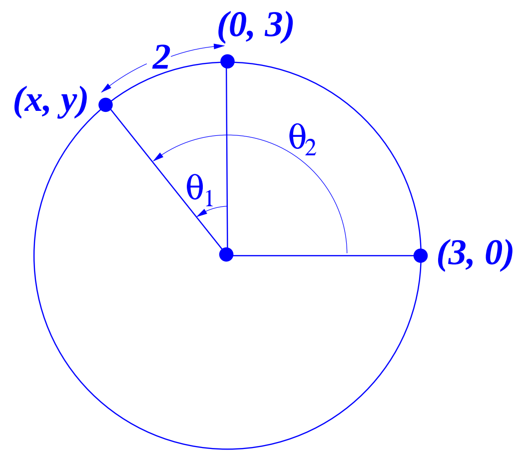
4. An ant starts at the point (0,3) on a circle of radius 3 (centered at the origin) and walks 2 units counterclockwise along the arc of the circle. Find the x and the y coordinates of where the ant ends up.

Referring to the diagram to the right, the ant starts at $(0,3)$ and walks a distance of 2 units counterclockwise to the point labeled (x,y). Since the angle θ_2 starts on the positive x-axis and terminates at the final position of the ant, we know that $x = 3\cos\theta_2$ and $y = 3\sin\theta_2$. Therefore, we need to find θ_2. To do this, first note that we can use $s = r\theta_1$ to find the value of θ_1:

$$s = r\theta_1 \implies 2 = 3\theta_1,$$

which implies $\theta_1 = 2/3$ radian. Therefore, $\theta_2 = (2/3) + (\pi/2)$ radians, and so we have

$$x = 3\cos\left(\frac{2}{3} + \frac{\pi}{2}\right) \approx -1.855 \text{ and } y = 3\sin\left(\frac{2}{3} + \frac{\pi}{2}\right) \approx 2.358.$$



Section 7.3 Supplement

Preliminary Example. Use the unit circles and corresponding triangles below to find the exact value of the sine and cosine of the special angles 30°, 45°, and 60°.

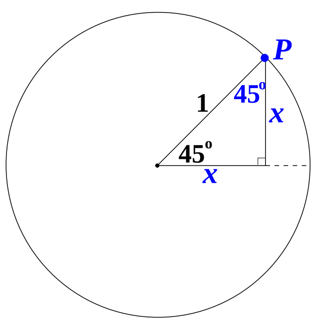


Figure 1.

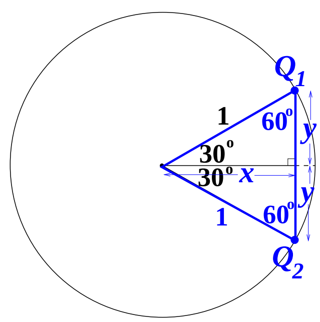


Figure 2.

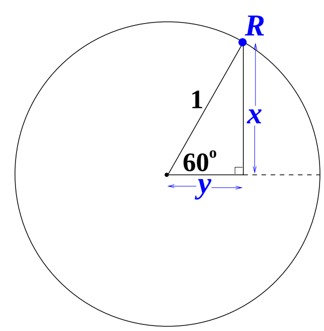


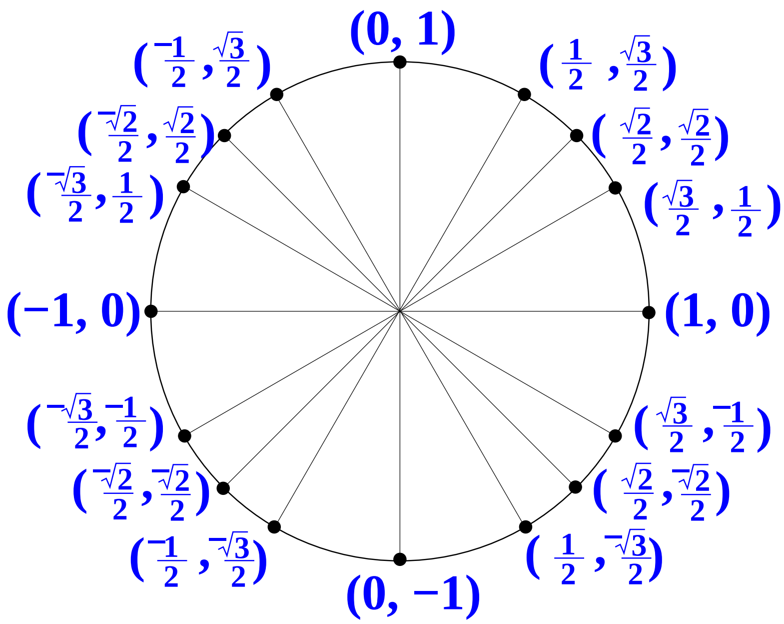
Figure 3.

Figure 1: Note first that the given triangle is an isosceles right triangle with hypotenuse 1, so we begin by letting x represent the length of each side of the triangle. Then we have $x^2 + x^2 = 1^2 \implies 2x^2 = 1 \implies x^2 = 1/2$, which means that $x = \sqrt{1/2}$, which can also be written as $x = \sqrt{2}/2$. It follows that $P = (\sqrt{2}/2,\ \sqrt{2}/2)$, and we conclude that $\cos 45° = \dfrac{\sqrt{2}}{2}$ and $\sin 45° = \dfrac{\sqrt{2}}{2}$.

Figure 2: Note first that the blue triangle is isosceles since two of its sides are radii of the circle and therefore both have length 1. Therefore, the angles opposite these legs have the same measure, and therefore must both measure 60° in order for the angles in the blue triangle to add up to 180°. This means that the blue triangle is equilateral, and so $\overline{Q_1Q_2} = 1$. But we can also observe that the two smaller triangles that make up the blue triangle are congruent by SAS; therefore, $y = 1/2$, and by using Pythagorean Theorem, we have $x^2 + (1/2)^2 = 1^2$, which implies $x^2 = 3/4$ and $x = \sqrt{3}/2$. Thus, $Q_1 = (\sqrt{3}/2,\ 1/2)$, and we conclude that $\cos 30° = \dfrac{\sqrt{3}}{2}$ and $\sin 30° = \dfrac{1}{2}$.

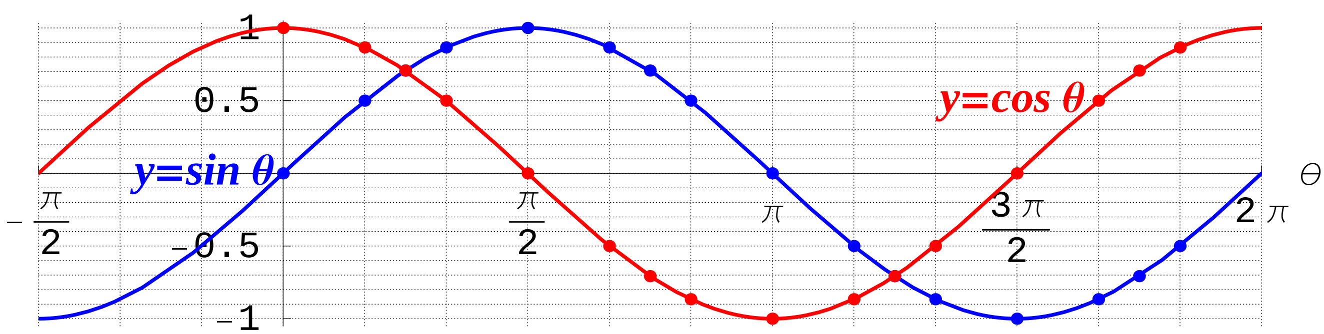
Figure 3: Note that the right triangle in Figure 3 shares the same interior angles and hypotenuse as each of the smaller triangles in Figure 2. Therefore, the right triangle in Figure 3 is congruent to each of the smaller triangles in Figure 2. We therefore conclude that the numbers x and y agree with the values from the Figure 2 analysis above. Therefore, $R = (y, x) = (1/2,\ \sqrt{3}/2)$, which means that $\cos 60° = \dfrac{1}{2}$ and $\sin 60° = \dfrac{\sqrt{3}}{2}$.

The Unit Circle



θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
θ	$0°$	$30°$	$45°$	$60°$	$90°$	$120°$	$135°$	$150°$	$180°$
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

θ	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
θ	$210°$	$225°$	$240°$	$270°$	$300°$	$315°$	$330°$	$360°$
$\cos\theta$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\sin\theta$	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0



Sections 7.4 and 7.5 – Sinusoidal Functions

Directions. Make sure that your graphing calculator is set in radian mode.

Function	Effect on $y = \sin x$
$y = 2\sin x$	vertical stretch by 2
$y = \sin x + 2$	shift up by 2
$y = \sin(x + 2)$	shift left by 2
$y = \sin(2x)$	horiz. compression by 2

B	$y = \sin(Bx)$	Period
1	$y = \sin x$	2π
2	$y = \sin(2x)$	π
4	$y = \sin(4x)$	$\pi/2$
1/2	$y = \sin(x/2)$	4π
B	$y = \sin(Bx)$	$2\pi/\lvert B\rvert$

Observation: $|B| \cdot \text{Period} = 2\pi$

Summary

For the sinusoidal functions $y = A\sin(B(x - h)) + k$ and $y = A\cos(B(x - h)) + k$:

1. Amplitude = $|A|$
2. Period = $\dfrac{2\pi}{|B|}$
3. Horizontal Shift = h
4. Midline: $y = k$

Definition. A function is called *sinusoidal* if it is a transformation of a sine or a cosine function.

Primary Goal in Section 7.5. Find formulas for sinusoidal functions given graphs, tables, or verbal descriptions of the functions.

Helpful Hints in Finding Formulas for Sinusoidal Functions

1. If selected starting point occurs at the midline of the graph, use the sine function.
2. If selected starting point occurs at the maximum or minimum value of the graph, use the cosine function.
3. Changing the sign of the constant "A" reflects the graph of a sinusoidal function about its midline.

Example 1. Let $y = 2\sin(2x - \pi) + 2$. Find the amplitude, period, midline, and horizontal shift of this function.

First, we rewrite the function as follows:

$$y = 2\sin(2x - \pi) + 2 \quad = \quad 2\sin(2(x - \frac{\pi}{2})) + 2$$

In this new form, we see that $A = 2$, $h = \pi/2$, $k = 2$, and $B = 2$. Therefore, our answers are as follows:

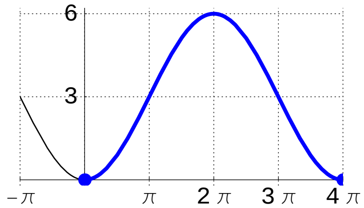
$$\text{Amplitude} = |A| = 2$$

$$\text{Horizontal Shift} = h = \frac{\pi}{2}$$

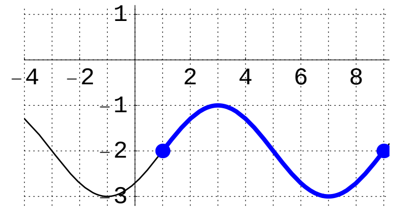
$$\text{Midline:} \quad y = k = 2$$

$$\text{Period} = \frac{2\pi}{|B|} = \pi$$

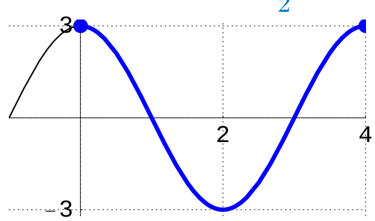
1. Find a possible formula for each of the following sinusoidal functions.



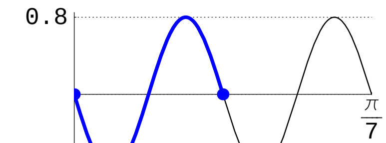
Referring to the highlighted portion of the graph above, we first note that the midline is $y = 3$ and the period is 4π. Therefore, $k = 3$ and $2\pi/B = 4\pi \implies B = 1/2$. Also, the amplitude is $(6 - 0)/2 = 3$, and since the highlighted portion of the graph is shaped like $-\cos x$, we have $A = -3$. Therefore, our answer is $y = -3\cos(\frac{1}{2}x) + 3$.



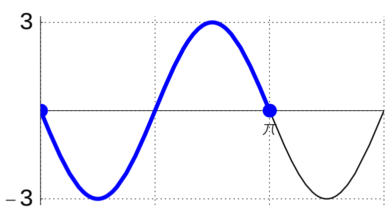
Using the same reasoning as to the left, $k = -2$ and the period is $7 - (-1) = 8$, meaning that $2\pi/B = 8$, which means that $B = \pi/4$. In this case, the highlighted portion of the graph looks like $\sin x$ shifted to the right $h = 1$ units. Our answer is therefore $y = \sin(\frac{\pi}{4}(x - 1)) - 2$.



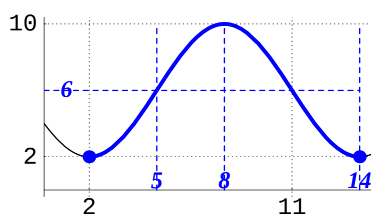
Here, we see that $k = 0$ and the period is 4, so that $2\pi/B = 4$, which means that $B = \pi/2$. Also, the highlighted portion of the graph is shaped like $y = \cos x$ with an amplitude of 3. Therefore, our answer is $y = 3\cos(\frac{\pi}{2}x)$.



Once again, we have $k = 0$, and we can see that the period is half of $\pi/7$, which is $\pi/14$. Thus, $2\pi/B = \pi/14$, which means that $B = 28$. Since the graph is shaped like $y = -\sin x$ with an amplitude of 0.8, our answer is $y = -0.8\sin(28x)$.



Again, $k = 0$ and we see that the period is π, which means that $2\pi/B = \pi \implies B = 2$. Since the highlighted portion looks like $y = -\sin x$ with an amplitude of 3, our answer is $y = -3\sin(2x)$.



Here, we have $k = 6$ and a period of $14 - 2 = 12$, which means that $B = \pi/6$. Since the highlighted portion looks like the graph of $y = -\cos x$ with an amplitude of 4, and shifted to the right by $h = 2$, our answer is $y = -4\cos(\frac{\pi}{6}(x - 2)) + 6$.

2. For each of the following, find the amplitude, the period, the horizontal shift, and the midline.

   (a) $y = 2\cos(\pi x + \frac{2\pi}{3}) - 1 = 2\cos(\pi(x - (-\frac{2}{3}))) - 1$

   From above, we see that $A = 2$, $k = -1$, $h = -2/3$, and $B = \pi$. Therefore, the amplitude is 2, the horizontal shift is $-2/3$, the midline is $y = -1$, and the period is $2\pi/\pi = 2$.

   (b) $y = 3 - \sin(2x - 7\pi) = -\sin(2(x - \frac{7\pi}{2})) + 3$

   From above, we see that $A = -1$, $k = 3$, $h = 7\pi/2$, and $B = 2$. Therefore, the amplitude is 1, the horizontal shift is $7\pi/2$, the midline is $y = 3$, and the period is $2\pi/2 = \pi$.

3. A population of animals oscillates annually from a low of 1300 on January 1st to a high of 2200 on July 1st, and back to a low of 1300 on the following January. Assume that the population is well-approximated by a sine or a cosine function.

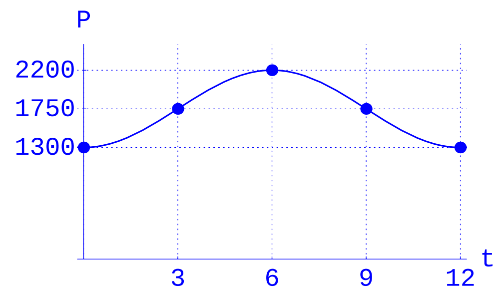
   (a) Find a formula for the population, P, as a function of time, t. Let t represent the number of months after January 1st. (**Hint.** First, make a rough sketch of the population, and use the sketch to find the amplitude, period, and midline.)

   Using our sketch to the right, we first note that the period is 12 months, so we know that

$$\frac{2\pi}{B} = 12 \quad \Longrightarrow \quad B = \frac{\pi}{6}.$$

   Also, the midline is the average of the maximum and minimum values of the function, so

$$k = \frac{1300 + 2200}{2} = 1750.$$



   Finally, the graph is shaped like $y = -\cos x$ with an amplitude of $(2200 - 1300)/2 = 450$, so $A = -450$. Our final answer is therefore

$$P = -450\cos(\frac{\pi}{6}t) + 1750.$$

   (b) Estimate the animal population on May 15th.

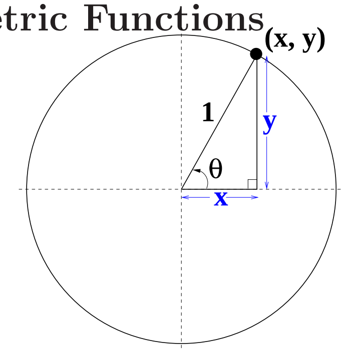
   Since $t = 0$ corresponds to January 1st, $t = 4$ corresponds to the beginning of May, so that $t = 4.5$ corresponds to May 15th. Therefore, our answer is

$$P = -450\cos(\frac{\pi}{6} \cdot 4.5) + 1750 \approx 2068 \text{ animals.}$$

   (c) On what dates will the animal population be halfway between the maximum and the minimum populations?

   This will occur when $t = 3$ and $t = 9$, which corresponds to April 1st and October 1st.

Section 7.6 – Tangent and Other Trigonometric Functions



$\cos\theta = x$

$\sin\theta = y$

$\tan\theta = \dfrac{y}{x}$ $\qquad$ $\cot\theta = \dfrac{x}{y}$

$\sec\theta = \dfrac{1}{x}$ $\qquad$ $\csc\theta = \dfrac{1}{y}$

From the picture, we have $x^2 + y^2 = 1^2$, which leads to the identity

$$\cos^2\theta + \sin^2\theta = 1.$$

Example 1. Suppose that $\cos\theta = \frac{2}{5}$ and that θ is in the 4th quadrant. Find $\sin\theta$ and $\tan\theta$ exactly.

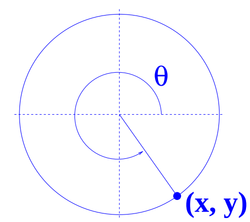
We have

$$\cos^2\theta + \sin^2\theta = 1 \implies \left(\frac{2}{5}\right)^2 + \sin^2\theta = 1 \implies \sin^2\theta = 1 - \frac{4}{25} \implies \sin^2\theta = \frac{21}{25}$$

$$\implies \sin\theta = \pm\frac{\sqrt{21}}{5}.$$

Since θ is a 4th quadrant angle, the y-coordinate associated with the terminal side of θ is negative (see diagram to the right), so

$$\sin\theta = -\sqrt{21}/5.$$



Therefore,

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-\sqrt{21}}{5}\cdot\frac{5}{2} = -\frac{\sqrt{21}}{2}.$$

Example 2. Find <u>exact</u> values for each of the following:

(a) $\tan\left(\frac{\pi}{6}\right)$ $\qquad$ (b) $\tan\left(\frac{\pi}{4}\right)$ $\qquad$ (c) $\tan\left(\frac{\pi}{2}\right)$

(a) $\tan\left(\dfrac{\pi}{6}\right) = \dfrac{\sin\left(\frac{\pi}{6}\right)}{\cos\left(\frac{\pi}{6}\right)} = \dfrac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \dfrac{1}{2}\cdot\dfrac{2}{\sqrt{3}} = \dfrac{1}{\sqrt{3}}$, or $\dfrac{\sqrt{3}}{3}$.

(b) $\tan\left(\dfrac{\pi}{4}\right) = \dfrac{\sin\left(\frac{\pi}{4}\right)}{\cos\left(\frac{\pi}{4}\right)} = \dfrac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1.$

(c) Since

$$\tan\left(\frac{\pi}{2}\right) = \frac{\sin\left(\frac{\pi}{2}\right)}{\cos\left(\frac{\pi}{2}\right)},$$

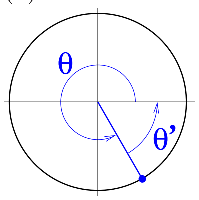
and since $\cos(\pi/2) = 0$, we see that $\tan(\pi/2)$ is undefined.

Reference Angles

Definition. The *reference angle* associated with an angle θ is the acute angle (having positive measure) formed by the x-axis and the terminal side of the angle θ.

Example. For each of the following angles, sketch the angle and find the reference angle.

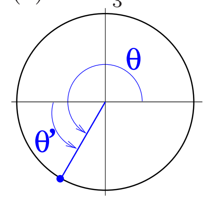
(1) $\theta = 300^\circ$



$\theta' = 360^\circ - 300^\circ = 60^\circ$, so

$$\cos 300^\circ = \cos 60^\circ = \frac{1}{2}$$
$$\sin 300^\circ = -\sin 60^\circ = -\frac{\sqrt{3}}{2}$$

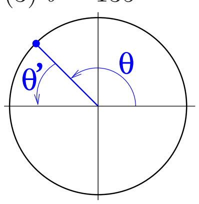
(2) $\theta = \frac{4\pi}{3}$



$\theta' = \frac{4\pi}{3} - \pi = \frac{\pi}{3}$, so

$$\cos \frac{4\pi}{3} = -\cos \frac{\pi}{3} = -\frac{1}{2}$$
$$\sin \frac{4\pi}{3} = -\sin \frac{\pi}{3} = -\frac{\sqrt{3}}{2}$$

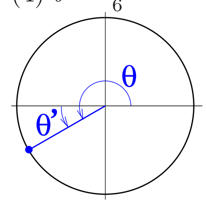
(3) $\theta = 135^\circ$



$\theta' = 180^\circ - 135^\circ = 45^\circ$, so

$$\cos 135^\circ = -\cos 45^\circ = -\frac{\sqrt{2}}{2}$$
$$\sin 135^\circ = \sin 45^\circ = \frac{\sqrt{2}}{2}$$

(4) $\theta = \frac{7\pi}{6}$



$\theta' = \frac{7\pi}{6} - \pi = \frac{\pi}{6}$, so

$$\cos \frac{7\pi}{6} = -\cos \frac{\pi}{6} = -\frac{\sqrt{3}}{2}$$
$$\sin \frac{7\pi}{6} = -\sin \frac{\pi}{6} = -\frac{1}{2}$$

Key Fact. If θ is any angle and θ' is the reference angle, then

$$\sin\theta' = \pm\sin\theta \qquad \tan\theta' = \pm\tan\theta \qquad \sec\theta' = \pm\sec\theta$$
$$\cos\theta' = \pm\cos\theta \qquad \csc\theta' = \pm\csc\theta \qquad \cot\theta' = \pm\cot\theta,$$

where the correct sign must be chosen based on the quadrant of the angle θ.

Exercise. Return to the previous example and find the exact value of the sine and the cosine of each angle.

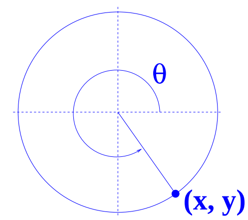
Examples and Exercises

1. Suppose that $\sin\theta = -\frac{3}{4}$ and that $\frac{3\pi}{2} \le \theta \le 2\pi$. Find the **exact values** of $\cos\theta$ and $\sec\theta$.

   We have

$$\sin^2\theta + \cos^2\theta = 1 \implies \left(-\frac{3}{4}\right)^2 + \cos^2\theta = 1 \implies \frac{9}{16} + \cos^2\theta = 1 \implies \cos^2\theta = \frac{7}{16}$$
$$\implies \cos\theta = \pm\frac{\sqrt{7}}{4}$$

   Since θ is a 4th quadrant angle, the x-coordinate associated with the terminal side of θ is positive (see diagram to the right), so $\cos\theta$ is positive. Therefore, our answers are:



$$\cos\theta = \frac{\sqrt{7}}{4}$$
$$\sec\theta = \frac{1}{\cos\theta} = \frac{4}{\sqrt{7}}, \text{ or } \frac{4\sqrt{7}}{7}$$

2. Suppose that $\csc\theta = \frac{x}{2}$ and that θ lies in the 2nd quadrant. Find expressions for $\cos\theta$ and $\tan\theta$ in terms of x.

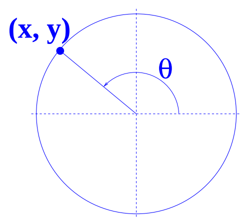
   First, note that

$$\csc\theta = \frac{x}{2} \implies \frac{1}{\sin\theta} = \frac{x}{2} \implies \sin\theta = \frac{2}{x}.$$

   Therefore, we have

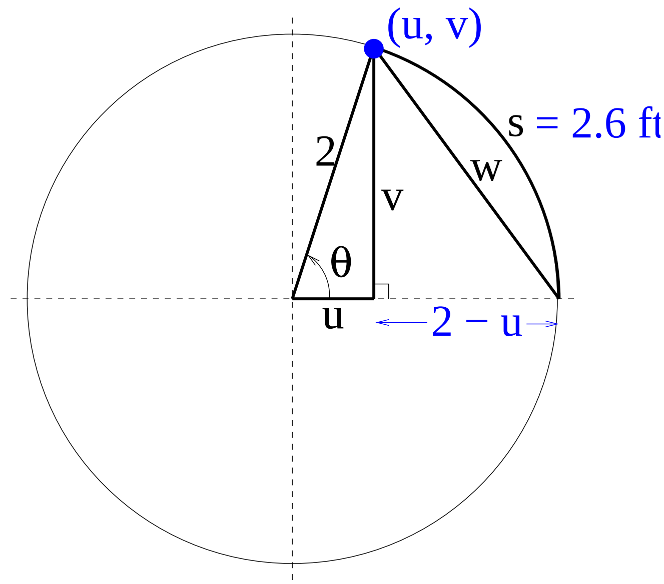
$$\sin^2\theta + \cos^2\theta = 1 \implies \left(\frac{2}{x}\right)^2 + \cos^2\theta = 1 \implies \frac{4}{x^2} + \cos^2\theta = 1 \implies \cos^2\theta = 1 - \frac{4}{x^2}$$
$$\implies \cos^2\theta = \frac{x^2}{x^2} - \frac{4}{x^2}$$
$$\implies \cos\theta = \pm\frac{\sqrt{x^2-4}}{x}.$$

   Since θ is a 2nd quadrant angle, the x-coordinate associated with the terminal side of θ is negative (see diagram to the right), so $\cos\theta$ is negative. Therefore, our answers are:



$$\cos\theta = -\frac{\sqrt{x^2-4}}{x}$$
$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{2/x}{(-\sqrt{x^2-4})/x} = \frac{2}{x}\cdot\frac{x}{-\sqrt{x^2-4}} = -\frac{2}{\sqrt{x^2-4}}$$

3. Given to the right is a circle of radius 2 feet (not drawn to scale). The length of the circular arc s is 2.6 feet. Find the lengths of the segments labeled u, v, and w. Give all answers rounded to the nearest 0.001.



First, note that the arc cut off by the angle θ measures $s = 2.6$ feet. Therefore, we have

$$s = r\theta \implies 2.6 = 2\theta,$$

so we conclude that $\theta = 2.6/2 = 1.3$ radians. Thus,

$$u = 2\cos 1.3 = 0.535$$
$$v = 2\sin 1.3 = 1.927$$

Thus, by Pythagorean Theorem, we have

$$w^2 = (2-u)^2 + v^2 \implies w^2 = (2-0.535)^2 + 1.927^2 \implies w^2 = 5.860 \implies w = 2.421.$$

To summarize, our final answers are $u = 0.535$, $v = 1.927$, and $w = 2.421$.

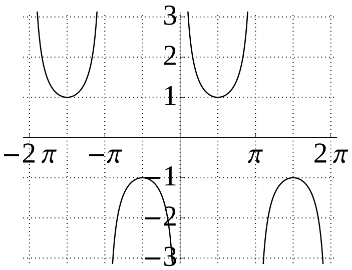
Section 7.7 – Trigonometric Relationships and Identities

Definitions. We define the *secant*, *cosecant*, and *cotangent* functions as follows:

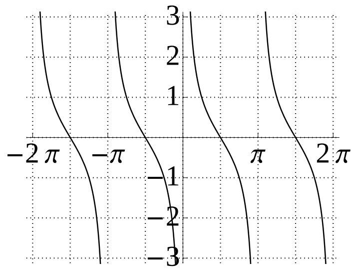
$$\sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Preliminary Exercise. Shown below are graphs of $y = \sec\theta$, $y = \csc\theta$, and $y = \cot\theta$ (not necessarily in that order). Without using a graphing calculator, match each graph to the correct formula. (**Suggestion:** Use your knowledge of the behavior of the sine and cosine functions to make your matches; in particular, pay attention to where these functions are positive, negative, or zero.)

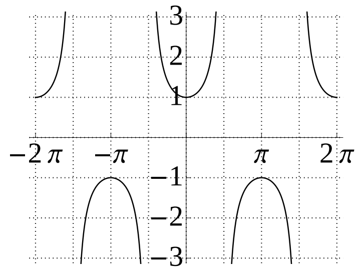
(a) $y = \csc\theta$



(b) $y = \cot\theta$



(c) $y = \sec\theta$

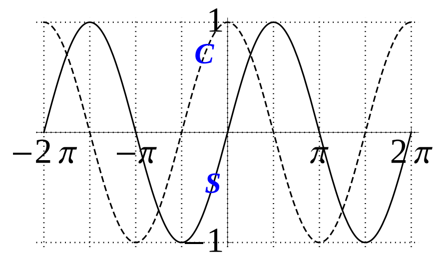


(a) Note that this graph is positive for $0 < \theta < \pi$ and negative for $\pi < \theta < 2\pi$, which agrees with the behavior for $y = \sin\theta$. Also, it has its asymptotes at multiples of π, which is where $\sin\theta$ equals 0. We conclude that (a) is the graph of $y = \csc\theta$.

(b) This graph must be the graph of $y = \cot\theta$ because its output values are 0 at $\pi/2$ and $3\pi/2$ (where $\cos\theta$, the numerator, is 0) and has asymptotes at multiples of π (where $\sin\theta$, the denominator, is 0).

(c) This graph has asymptotes where $\cos\theta = 0$ and is positive and negative on the same intervals as the cosine function, so we conclude that $y = \sec\theta$ matches this graph.

Example 1. Given to the right are graphs of the sine and cosine functions. First, label which is which and then complete the following:



(a) Describe how the cosine function can be translated to obtain the sine function, and vice versa. Then, use your observations to complete identities (1) and (2) below.

By moving the cosine function $\pi/2$ to the right, we obtain the sine function (see Identity 1), and by moving the sine function $\pi/2$ to the left, we obtain the cosine function (see Identity 2).

(b) Is the cosine function even, odd, or neither? How about the sine function? Use your observations to complete identities (3) and (4) below.

The cosine function is even (symmetry about the y-axis), and the sine function is odd (symmetry about the origin).

Identities.

(1) $\cos\left(\theta - \frac{\pi}{2}\right) = \sin\theta$

(2) $\sin\left(\theta + \frac{\pi}{2}\right) = \cos\theta$

(3) $\cos(-\theta) = \cos\theta$

(4) $\sin(-\theta) = -\sin\theta$

Example 2. Illustrate Identity (2) from page 17 by filling in the table to the right. What do you notice?

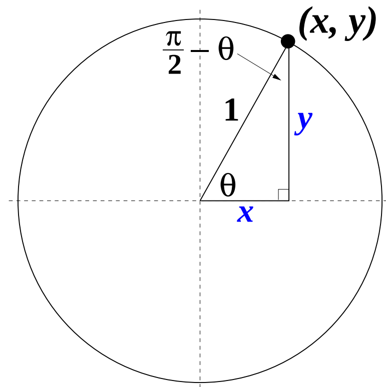
θ	$\cos\theta$	$\sin\left(\theta+\frac{\pi}{2}\right)$
0	1	$\sin(\pi/2) = 1$
$\pi/6$	$\sqrt{3}/2$	$\sin(2\pi/3) = \sqrt{3}/2$
$\pi/4$	$\sqrt{2}/2$	$\sin(3\pi/4) = \sqrt{2}/2$
$\pi/3$	$1/2$	$\sin(5\pi/6) = 1/2$
$\pi/2$	0	$\sin(\pi) = 0$

For all values of θ in the table, the output values of $\cos\theta$ and $\sin\left(\theta+\frac{\pi}{2}\right)$ are the same. In fact, for <u>any</u> value of θ, these two expressions will be equal. This makes sense, since our analysis from Example 1 demonstrates that $y = \cos\theta$ and $y = \sin\left(\theta+\frac{\pi}{2}\right)$ are identical functions.

More Identities

<u>Some observations</u>:

Since $x = \cos\theta$ and $y = \sin\theta$ in the diagram to the right, we have (by Pythagorean's Theorem):



$$x^2 + y^2 = 1^2 \implies \underline{(\cos\theta)^2 + (\sin\theta)^2 = 1}$$

$$\left.\begin{aligned}\sin\theta &= \frac{y}{1}\\ \cos\left(\frac{\pi}{2}-\theta\right) &= \frac{y}{1}\end{aligned}\right\} \qquad \underline{\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta}$$

$$\left.\begin{aligned}\cos\theta &= \frac{x}{1}\\ \sin\left(\frac{\pi}{2}-\theta\right) &= \frac{x}{1}\end{aligned}\right\} \qquad \underline{\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta}$$

Identities.

(5) $\sin^2\theta + \cos^2\theta = 1$ (6) $\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$ (7) $\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta$

Example 3. Given that $\sin\theta = \frac{1}{3}$ and that θ is in the second quadrant, find the <u>exact</u> values of $\cos\theta$, $\tan\theta$, and $\sec\theta$.

Using Identity (5) from above, we have

$$\begin{aligned}\sin^2\theta + \cos^2\theta = 1 \quad &\implies \quad \left(\frac{1}{3}\right)^2 + \cos^2\theta = 1\\ &\implies \quad \cos^2\theta = 1 - \frac{1}{9}\\ &\implies \quad \cos\theta = \pm\sqrt{\frac{8}{9}} = \pm\frac{\sqrt{8}}{3}\end{aligned}$$

Now, since θ is in the 2nd quadrant, we know that $\cos\theta$ is negative, so we conclude that

$$\cos\theta = -\frac{\sqrt{8}}{3} = -\frac{2\sqrt{2}}{3}.$$

Therefore, we have

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{1/3}{(-2\sqrt{2})/3} = \frac{1}{3}\cdot\left(-\frac{3}{2\sqrt{2}}\right) = -\frac{1}{2\sqrt{2}}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{(-2\sqrt{2})/3} = \frac{1}{1}\cdot\left(-\frac{3}{2\sqrt{2}}\right) = -\frac{3}{2\sqrt{2}}$$

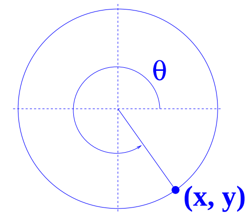
Examples and Exercises

1. Suppose that $\sin\theta = -\frac{3}{4}$ and that $\frac{3\pi}{2} \le \theta \le 2\pi$. Find the **exact values** of $\cos\theta$ and $\sec\theta$.

We have

$$\sin^2\theta + \cos^2\theta = 1 \implies \left(-\frac{3}{4}\right)^2 + \cos^2\theta = 1 \implies \frac{9}{16} + \cos^2\theta = 1 \implies \cos^2\theta = \frac{7}{16}$$
$$\implies \cos\theta = \pm\frac{\sqrt{7}}{4}$$

Since θ is a 4th quadrant angle, the x-coordinate associated with the terminal side of θ is positive (see diagram to the right), so $\cos\theta$ is positive. Therefore, our answers are:



$$\cos\theta = \frac{\sqrt{7}}{4}$$
$$\sec\theta = \frac{1}{\cos\theta} = \frac{4}{\sqrt{7}}, \text{ or } \frac{4\sqrt{7}}{7}$$

2. Suppose that $\csc\theta = \frac{x}{2}$ and that θ lies in the 2nd quadrant. Find expressions for $\cos\theta$ and $\tan\theta$ in terms of x.

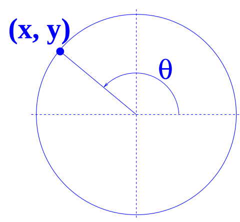
First, note that

$$\csc\theta = \frac{x}{2} \implies \frac{1}{\sin\theta} = \frac{x}{2} \implies \sin\theta = \frac{2}{x}.$$

Therefore, we have

$$\sin^2\theta + \cos^2\theta = 1 \implies \left(\frac{2}{x}\right)^2 + \cos^2\theta = 1 \implies \frac{4}{x^2} + \cos^2\theta = 1 \implies \cos^2\theta = 1 - \frac{4}{x^2}$$
$$\implies \cos^2\theta = \frac{x^2}{x^2} - \frac{4}{x^2}$$
$$\implies \cos\theta = \pm\frac{\sqrt{x^2-4}}{x}.$$

Since θ is a 2nd quadrant angle, the x-coordinate associated with the terminal side of θ is negative (see diagram to the right), so $\cos\theta$ is negative. Therefore, our answers are:



$$\cos\theta = -\frac{\sqrt{x^2-4}}{x}$$
$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{2/x}{(-\sqrt{x^2-4})/x} = \frac{2}{x}\cdot\frac{x}{-\sqrt{x^2-4}} = -\frac{2}{\sqrt{x^2-4}}$$

Section 7.8 – Inverse Trigonometric Functions

Preliminary Idea.

$$\sin(\pi/6) = 1/2 \qquad \text{means the same thing as} \qquad \underline{\sin^{-1}(1/2) = \pi/6}.$$

Definition.

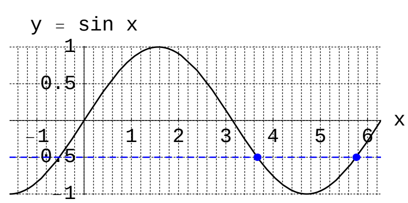
1. $\sin^{-1} x$ is the angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ whose sine is x.
2. $\tan^{-1} x$ is the angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ whose tangent is x.
3. $\cos^{-1} x$ is the angle between 0 and π whose cosine is x.

Note. "$\sin^{-1} x$", "$\cos^{-1} x$", and "$\tan^{-1} x$" can also be written as "$\arcsin x$", "$\arccos x$", and "$\arctan x$", respectively.

Example 1. Calculate each of the following exactly.

1. $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \underline{\frac{\pi}{6}}$ (since $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$)
2. $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \underline{\frac{\pi}{4}}$ (since $\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$)
3. $\tan^{-1}(\sqrt{3}) = \underline{\frac{\pi}{3}}$
4. $\sin^{-1}(-1) = \underline{-\frac{\pi}{2}}$

Example 2. Use the graph to estimate, to the nearest 0.1, all solutions to the equation $\sin x = -\frac{1}{2}$ that lie between 0 and 2π. Then, find the solutions exactly.

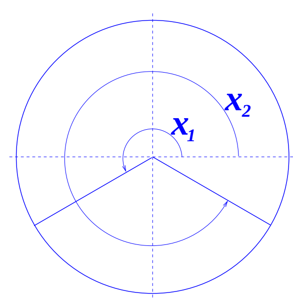


By estimating where the graph of $y = \sin x$ crosses the line $y = -0.5$, we see that the approximate values of the solutions are $x_1 = 3.7$ and $x_2 = 5.8$.

To find the exact values, we use the unit circle and reference angles. Since

$$\sin x' = \frac{1}{2} \implies x' = \frac{\pi}{6},$$

we see that the reference angle is $x' = \pi/6$. Since the sine function is negative in the third and fourth quadrants, our solutions are the angles x_1 and x_2 illustrated in the diagram to the right. Our answers are therefore:



$$x_1 = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$$

$$x_2 = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$$

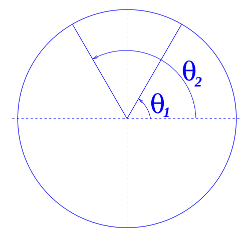
Note that $7\pi/6 \approx 3.7$ and $11\pi/6 \approx 5.8$, so our exact answers agree with our graphical approximations.

Examples and Exercises

1. Solve each of the following trigonometric equations, giving all solutions between 0 and 2π. Give **exact** answers whenever possible.

   (a) $\sin\theta = \frac{\sqrt{3}}{2}$ Reference Angle: $\sin\theta' = \frac{\sqrt{3}}{2} \implies \theta' = \frac{\pi}{3}$

   Since the sine function is positive in the first and second quadrants, we have:

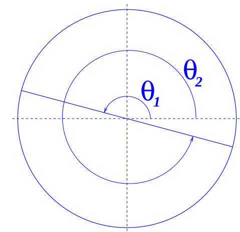
   

   $$\theta_1 = \frac{\pi}{3}$$
   $$\theta_2 = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

   Therefore, our answer is $\{\frac{\pi}{3},\ \frac{2\pi}{3}\}$.

   (b) $\tan\theta = -0.3$ Reference Angle: $\tan\theta' = 0.3 \implies \theta' = \tan^{-1}(0.3) \approx 0.291$

   Since the tangent function is negative in the second and fourth quadrants, we have:

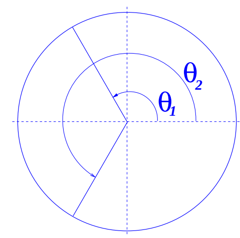
   

   $$\theta_1 = \pi - 0.291 \approx 2.850$$
   $$\theta_2 = 2\pi - 0.291 \approx 5.992$$

   Therefore, our answer is $\{2.850,\ 5.992\}$.

   (c) $\cos\theta = -\frac{1}{2}$ Reference Angle: $\cos\theta' = \frac{1}{2} \implies \theta' = \frac{\pi}{3}$

   Since the cosine function is negative in the second and third quadrants, we have:

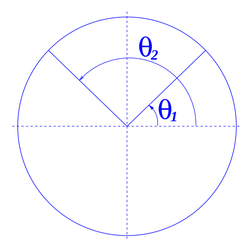
   

   $$\theta_1 = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$
   $$\theta_2 = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$$

   Therefore, our answer is $\{\frac{2\pi}{3},\ \frac{4\pi}{3}\}$.

   (d) $\sin\theta = 0.7$ Reference Angle: $\sin\theta' = 0.7 \implies \theta' = \sin^{-1}(0.7) \approx 0.775$

   Since the sine function is positive in the first and second quadrants, we have:

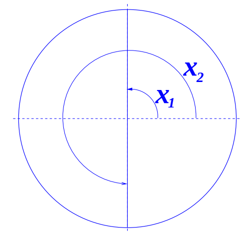
   $$\theta_1 \approx 0.775$$
   $$\theta_2 = \pi - 0.775 \approx 2.366$$

   Therefore, our answer is $\{0.775,\ 2.366\}$.

2. Find all solutions to $2\sin x\cos x + \cos x = 0$ that lie between 0 and 2π. Give your answers **exactly**.

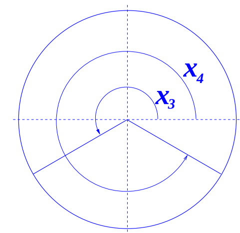
$$2\sin x\cos x + \cos x = 0 \implies \cos x(2\sin x + 1) = 0 \implies \cos x = 0 \text{ or } 2\sin x + 1 = 0$$
$$\implies \cos x = 0 \text{ or } \sin x = -\frac{1}{2}$$

If $\cos x = 0$, then our solutions are



$$x_1 = \frac{\pi}{2}$$
$$x_2 = \frac{3\pi}{2}$$

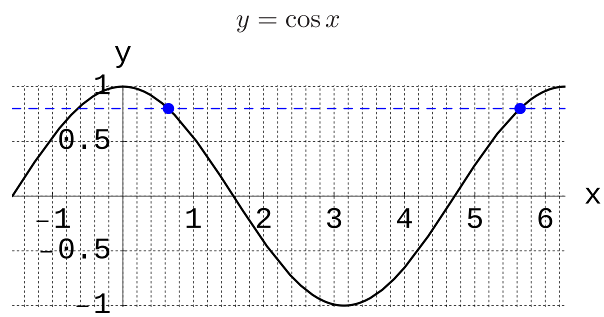
On the other hand, if $\sin x = -1/2$, then our reference angle is $x' = \pi/6$, and we have solutions in the 3rd and 4th quadrant:



$$x_3 = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$$
$$x_4 = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$$

Therefore, our final answer is $\{\frac{\pi}{2},\ \frac{3\pi}{2},\ \frac{7\pi}{6},\ \frac{11\pi}{6}\}$.

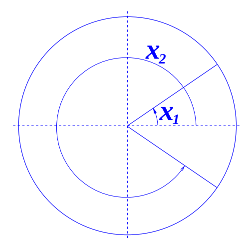
3. Use the graph to the right to estimate the solutions to the equation $\cos x = 0.8$ that lie between 0 and 2π. Then, use reference angles to find more accurate estimates of your solutions.



From the graph to the right, we see that the two solutions between 0 and 2π are

$$x_1 \approx 0.6 \text{ and } x_2 \approx 5.6.$$

To find more accurate answers, we use reference angles. Note that in this case, the reference angle is $x' = \cos^{-1}(0.8) \approx 0.644$. Because the cosine function is positive in the first and fourth quadrants, our solutions are:

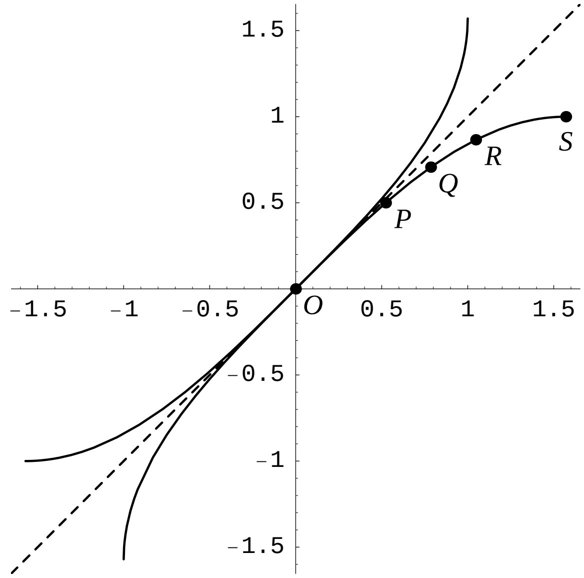


$$x_1 \approx 0.644$$
$$x_2 = 2\pi - 0.644 \approx 5.640$$

Therefore, our answer is $\{0.644,\ 5.640\}$.

x	$\sin x$
0	0
$\pi/6$	$1/2$
$\pi/4$	$\sqrt{2}/2$
$\pi/3$	$\sqrt{3}/2$
$\pi/2$	1

x	$\sin^{-1} x$
0	0
$1/2$	$\pi/6$
$\sqrt{2}/2$	$\pi/4$
$\sqrt{3}/2$	$\pi/3$
1	$\pi/2$



x	$\tan x$
0	0
$\pi/6$	$\sqrt{3}/3$
$\pi/4$	1
$\pi/3$	$\sqrt{3}$
$\pi/2$	undefined

x	$\tan^{-1} x$
0	0
$\sqrt{3}/3$	$\pi/6$
1	$\pi/4$
$\sqrt{3}$	$\pi/3$
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